MATH 1010 E Week 9 Lecture Notes (Martin Li)

Last time ... Indefinite integral $\int f(x)\,dx$

as solution to $\boxed{F'(x) = f(x)}$

e.g. $$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C.$$

u - substitution

Guiding example: Evaluate $\int x\sqrt{x^2+4}\,dx$.

Sol: Let $u = x^2+4$.

$$\Rightarrow \frac{du}{dx} = 2x \Rightarrow \boxed{du = 2x\,dx} \quad \text{"differential".}$$

Substitution $\Rightarrow$ $$\int x\sqrt{x^2+4}\,dx = \int \left(\frac{1}{2}\sqrt{x^2+4}\right)(2x\,dx)$$

$$= \int \frac{1}{2}\sqrt{u}\,du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C$$

$$= \frac{1}{3}(x^2+4)^{3/2} + C$$

※

Q: Why can we do that ?

Philosophy: "a rule for differentiation" $\iff$ "a rule for integration".

① $\dfrac{d}{dx}[f(x) \pm g(x)] = \dfrac{df}{dx} \pm \dfrac{dg}{dx} \iff \int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

② $\dfrac{d}{dx}[k \cdot f(x)] = k \cdot \dfrac{df}{dx} \iff \int k \cdot f(x)\,dx = k \cdot \int f(x)\,dx$

③ product / quotient rules $\iff$? (later)

④ chain rule $\iff$? . (Ans: u-substitution!)

Chain Rule

$$\frac{d}{dx}\Big(\overbrace{f(u(x))}^{F(x)}\Big) = \overbrace{f'(u(x)) \cdot u'(x)}^{f(x)}$$

$$\Updownarrow \qquad \boxed{F'(x) = f(x)}$$

$$f(u(x)) = \int f'(u(x)) \cdot \underbrace{u'(x)\,dx}_{du = u'(x)\,dx} = \int f'(u)\,du$$

More Examples

(1) $$\int \frac{x\,dx}{(1+x^2)^2} = \int \frac{\frac{1}{2}du}{u^2} = \frac{1}{2}\int \frac{1}{u^2}du$$

Let $u = 1+x^2$

$du = 2x\,dx$

$$= \frac{1}{2}\frac{u^{-1}}{-1} + C = -\frac{1}{2}\frac{1}{1+x^2} + C$$

※

Bad substitution:

Let $u = (1+x^2)^2$, $du = 2(1+x^2)\cdot(2x)\,dx$

$\Downarrow$

$x\,dx = \frac{du}{4(1+x^2)}$

$= \frac{du}{4\sqrt{u}}$.

$$\int \frac{x\,dx}{(1+x^2)^2} = \int \frac{1}{u}\frac{du}{4\sqrt{u}}$$

$$= \frac{1}{4}\int u^{-3/2}\,du$$

$$= \frac{1}{4}\frac{u^{-1/2}}{-1/2} + C = -\frac{1}{2}\frac{1}{\sqrt{u}} + C$$

$$= -\frac{1}{2}\frac{1}{1+x^2} + C$$

※.

(2) $$\boxed{\int \tan x\,dx = ?}$$

Note: $$\int \tan x\,dx = \int \frac{\sin x}{\cos x}dx$$

Let $u = \cos x$

$du = -\sin x\,dx$

$$= \int \frac{-du}{u} = -\ln|u| + C$$

$$= -\ln|\cos x| + C$$

※.

Check: $$\frac{d}{dx} -\ln|\cos x| = -\frac{-\sin x}{\cos x} = \frac{\sin x}{\cos x} = \tan x.$$

(3). $\int x e^{-x^2} dx = \int \frac{1}{2} e^{-u} du = \frac{-1}{2} e^{-u} + C$

Let $u = x^2$

$du = 2x\, dx$

$= -\frac{1}{2} e^{-x^2} + C$ #

(4) $\int \frac{1}{1+x} dx = \int \frac{1}{1+x} d(1+x)$

$= \ln|1+x| + C$ #

(5) $\int \frac{x}{1+x} dx = \int \frac{(1+x)-1}{1+x} dx$

$= \int \left(1 - \frac{1}{1+x}\right) dx$

(partial fractions!)

$= x - \ln|1+x| + C$ #

(6) $\boxed{\int \sin^2 x\, dx = ?}$

$u = \sin x$

$du = \cos x\, dx$

$\int \sin^2 x\, dx = \int u^2 \frac{du}{\cos x}$

$= \int \frac{u^2}{\sqrt{1-\sin^2 x}} du$

$= \int \frac{u^2}{\sqrt{1-u^2}} du$

$= !!$

If the integral were

$\int \sin 2x\, dx$

$= \int \frac{1}{2} \sin(2x)\, d(2x)$

$= \frac{1}{2}(-\cos 2x) + C$

Recall: (Half - angle formula).

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\Rightarrow \int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} \, dx$$

$$= \frac{1}{2}\left[\int 1 \, dx - \int \cos 2x \, dx \right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2} \right] + C \quad \#$$

Ex: $\int \cos^2 x \, dx$ by (i) half - angle formula

(ii) using our result above.

$(\because \sin^2 x + \cos^2 x = 1)$

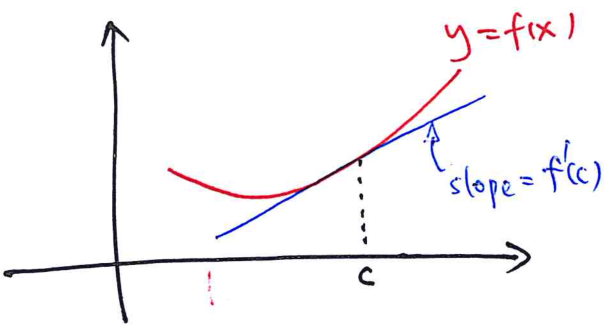
Definite Integrals

- comparison:

$\boxed{\text{differentiation}}$

$$f(x) \xrightarrow{\frac{d}{dx}} f'(x) \leftarrow \text{function}$$

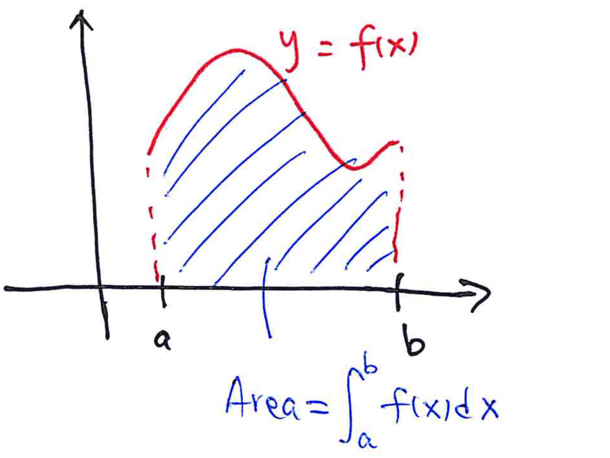
$$f(x) \xrightarrow{\left.\frac{d}{dx}\right|_{x=c}} f'(c) \leftarrow \text{number}$$



$\boxed{\text{Integration}}$

$$f(x) \xrightarrow{\int \cdot dx} \int f(x) dx \leftarrow \text{function}$$

$$f(x) \xrightarrow{\int_a^b \cdot dx} \int_a^b f(x) dx \leftarrow \text{number}$$



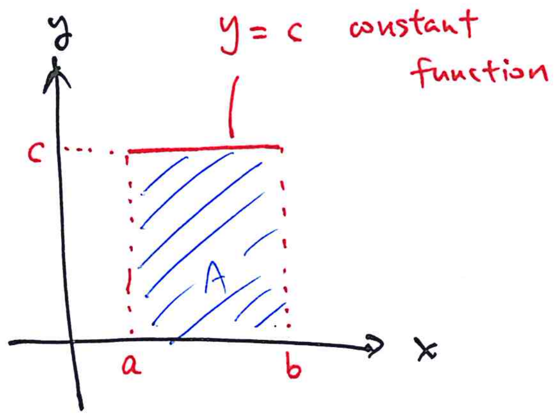
Problem (Geometric)

Given the graph $y = f(x)$ of a (continuous) function

$$f : [a, b] \longrightarrow \mathbb{R}$$

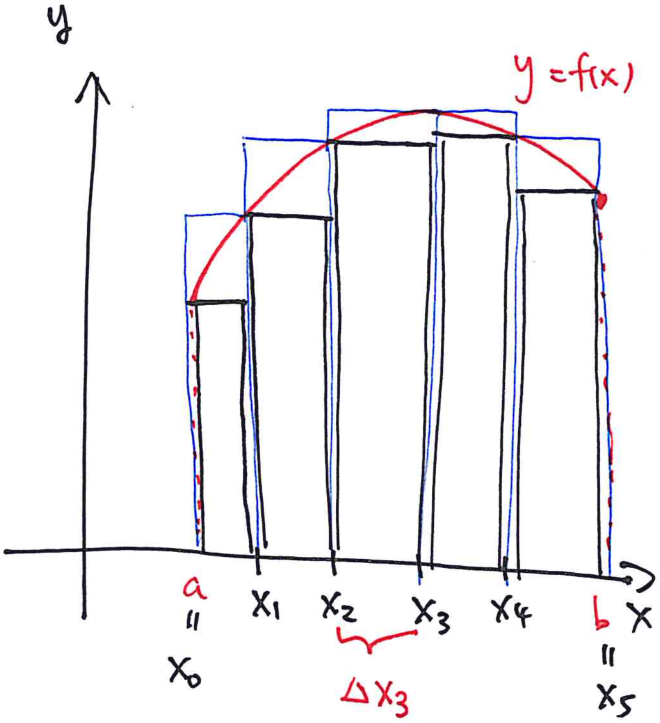
How to find the area under the graph?

Simplest case: "Rectangle"

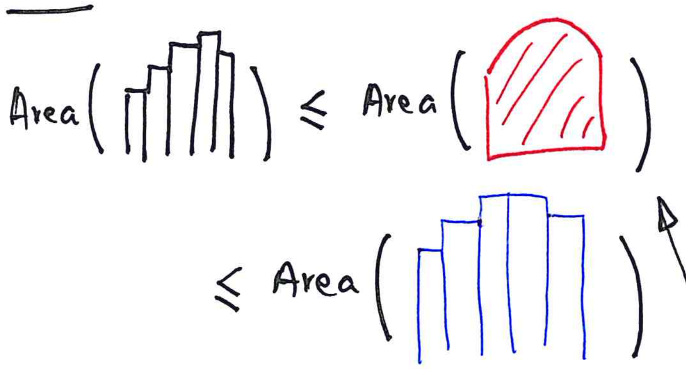


Area of this rectangle $= c(b-a)$.

General case: "Approximation by Rectangles" - Riemann integrals.



Idea:



$\Downarrow$ If "partition" is fine enough.

all the same number

$\parallel$

$$\int_a^b f(x)\,dx = \text{Area}$$

Do this more carefully

Given $f : [a,b] \to \mathbb{R}$ (say "continuous")

Step 1: take a partition P of $[a,b]$

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Call size of subinterval $[x_{k-1}, x_k]$, $k = 1, \dots, n$

$$\Delta x_k := x_k - x_{k-1}$$

and mesh of P : (measure of "fineness" of P)

$$|P| := \max_{k=1,\dots,n} \Delta x_k .$$

Step 2: On each subinterval $[x_{k-1}, x_k]$, define

$$M_k := \max_{x \in [x_{k-1}, x_k]} f(x)$$

$$m_k := \min_{x \in [x_{k-1}, x_k]} f(x)$$

Note: f cts $\Rightarrow$ M_k, m_k exists and achieved at some points

Step 3: Approximate by rectangles :

$$\mathcal{U}(f,P) := \sum_{k=1}^{n} M_k \cdot \Delta x_k \qquad \text{upper sum}$$

$$\mathcal{L}(f,P) := \sum_{k=1}^{n} m_k \cdot \Delta x_k \qquad \text{lower sum}$$

Clearly, $\mathcal{L}(f,P) \le \text{Area} \le \mathcal{U}(f,P)$

Step 4: Refine the partition s.t $|P| \to 0$

If $\lim_{|P| \to 0} \mathcal{U}(f,P) = \lim_{|P| \to 0} \mathcal{L}(f,P)$, then we say that f is Riemann integrable

"A"

and $\displaystyle\int_a^b f(x)\,dx := A$ definite integral of f

Thm: If $f:[a,b]\to\mathbb{R}$ is continuous

then f is Riemann integrable.

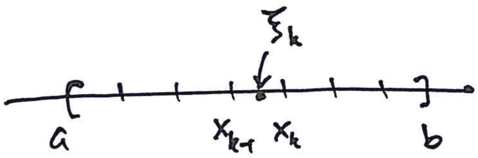
and $$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\,\Delta x_k\,.}$$ (Riemann sum)

where $\xi_k \in [x_{k-1}, x_k]$ is ANY point

and $$\left.\begin{aligned} x_k &= a + k\cdot\frac{b-a}{n} \\ \Delta x_k &= \frac{b-a}{n} \end{aligned}\right\}$$ even partition

Example:

(0) $\displaystyle\int_a^b 1\,dx = b-a$



$$\int_a^b 1\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\,\Delta x_k$$

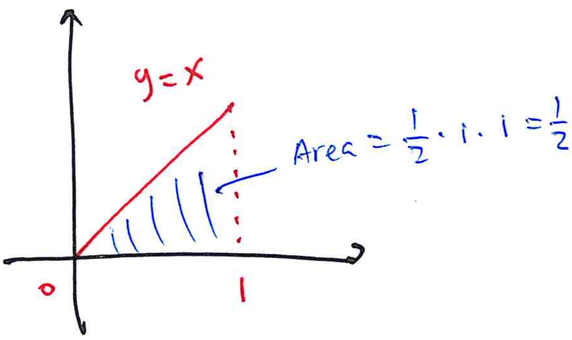
$$= \lim_{n\to\infty}\sum_{k=1}^{n} 1\,\Delta x_k$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{b-a}{n}$$

$$= \lim_{n\to\infty}\left(\frac{b-a}{\cancel{n}}\right)\cdot\cancel{n} = b-a$$ $\#$

(1) $$\int_0^1 x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f(\xi_k)\,\Delta x_k \qquad \xi_k \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

take $\xi_k = \frac{k-1}{n}$.



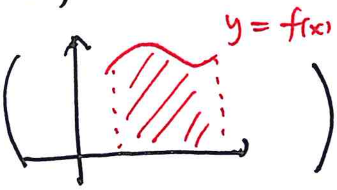
$$= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{k-1}{n}\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n^2}\underbrace{\sum_{k=0}^{n-1}k}_{} = \lim_{n\to\infty}\frac{n(n-1)}{2n^2} = \frac{1}{2} \; ※$$

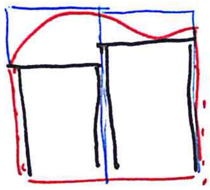
$$1+2+3+\cdots + (n-1)$$

$$= \frac{(n-1)n}{2}$$

Last time... u-substitution (chain rule)

Definite integrals : $\int_a^b f(x)\,dx$ = Area ()

Recall: $P: \quad x_0 < x_1 < x_2 < \cdots < x_n$, $x_0 = a$, $x_n = b$.



$$\mathcal{L}(f,P) \le \text{Area} \le \mathcal{U}(f,P)$$

Hope, as $|P| \to 0$ (finer partition)

$$\int_a^b f(x)\,dx := \lim_{|P|\to 0} \mathcal{L}(f,P) = \lim_{|P|\to 0} \mathcal{U}(f,P) \quad \text{if exists.}$$

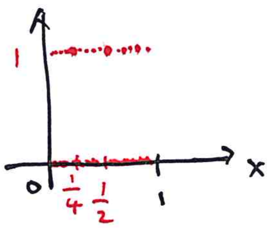
$\Downarrow$

f Riemann integrable.

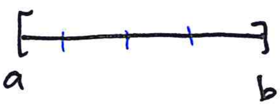
Fact: Some functions are NOT Riemann integrable.

Adv Q:

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1], \text{ rational ie } x = \frac{p}{q}. \\ 0 & \text{if } x \in [0,1], \text{ irrational.} \end{cases}$$



Thm: Any continuous function $f: [a,b] \to \mathbb{R}$ is Riemann integrable, moreover, we can compute it using "Riemann sum".



"uniform partition."

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)\cdot \frac{b-a}{n}}$$

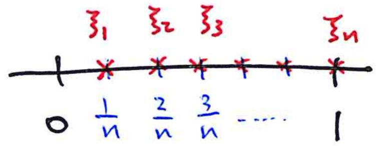
where $\xi_k \in [x_{k-1}, x_k]$ any

$P: \quad a = x_0 < x_1 = x_0 + \frac{b-a}{n} < \cdots < x_k = x_0 + k\left(\frac{b-a}{n}\right) < \cdots \le x_n = b.$

Examples: (Last time: $\int_a^b 1\,dx = b-a$; $\int_0^1 x\,dx = \frac{1}{2}$)

(1) $\int_0^1 x^2\,dx$. Let $f(x) = x^2$, $x \in [0,1]$.

$$\sum_{k=1}^{n} f(\xi_k)\Delta x_k = \sum_{k=1}^{n} f\left(\frac{k}{n}\right)\cdot\frac{1}{n}$$



$$= \sum_{k=1}^{n} \frac{k^2}{n^2}\cdot\frac{1}{n}$$

Fact: $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

$$= \frac{1}{n^3}\left(\sum_{k=1}^{n} k^2\right)$$

$$= \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6} \xrightarrow[n\to\infty]{} \frac{2}{6} = \frac{1}{3}.$$

$$\Rightarrow \quad \int_0^1 x^3\,dx = \frac{1}{3} \quad \#.$$

Note: You can take other choices of ξ_k.

(2) $$\int_0^1 e^x\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} e^{\frac{k}{n}}\cdot\frac{1}{n}$$

$$= \lim_{n\to\infty}\frac{1}{n}\Big(\underbrace{e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e^{\frac{n}{n}}}_{\text{geometric series}}\Big)$$

Note:
$a + ar + ar^2 + \cdots + ar^{n-1}$
$= \frac{a(1-r^n)}{1-r}$

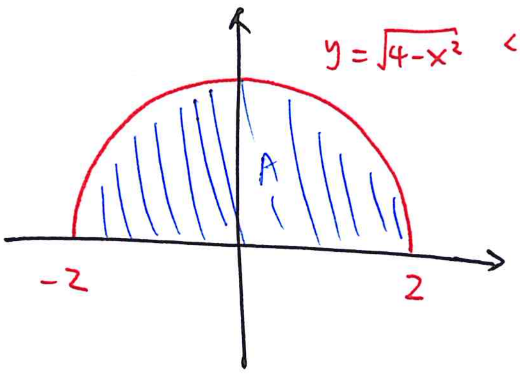
$$= \lim_{n\to\infty}\frac{1}{n}\,\frac{e^{\frac{1}{n}}(1-e^{\frac{n}{n}})}{1-e^{\frac{1}{n}}}$$

$$= (1-e)\lim_{n\to\infty}\frac{e^{\frac{1}{n}}}{n(1-e^{\frac{1}{n}})}$$

$$= (1-e)\lim_{n\to\infty}\overbrace{\frac{\frac{1}{n}e^{\frac{1}{n}}}{1-e^{\frac{1}{n}}}}^{-1}$$

$$= (1-e)\lim_{x\to 0}\frac{xe^x}{1-e^x} = (1-e)\lim_{x\to 0}\frac{e^x + xe^x}{-e^x} = e-1 \quad \#$$

(3) $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx = A = \frac{1}{2}\pi(2)^2 = 2\pi$ *



(Calculation from geometry.)

Properties of Definite Integrals

(1) $\displaystyle\int_a^b [f(x)+g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(2) $\displaystyle\int_a^b k\cdot f(x)\,dx = k\cdot\int_a^b f(x)\,dx \qquad k = \text{constant.}$

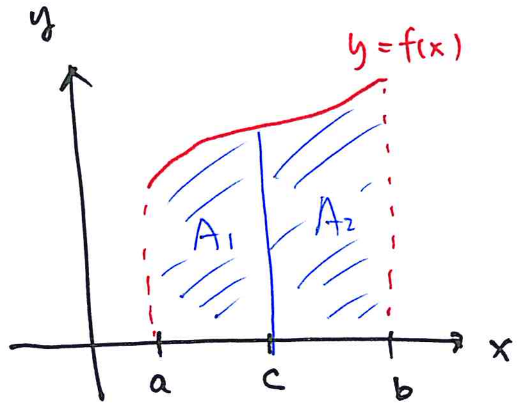
(linear

(3) If $a<b$, then

$$\int_b^a f(x)\,dx := -\int_a^b f(x)\,dx$$

E.g. $\displaystyle\int_1^0 x\,dx = -\int_0^1 x\,dx = -\frac{1}{2}$ *.

(4) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.



[Fact: this holds even when $c>b$ or $c<a$]

$A_1 + A_2 = A = \displaystyle\int_a^b f(x)\,dx$.

Special Example:

Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \frac{\pi}{4}.$

Recall: Symmetry: $\cos\left(\frac{\pi}{2} - x\right) = \sin x.$

Hint: Prove that

$$\int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

$\Rightarrow$ Since $\sin^2 x + \cos^2 x = 1$

$$\underbrace{\int_0^{\frac{\pi}{2}} \sin^2 x \, dx}_{= \frac{\pi}{4}} + \underbrace{\int_0^{\frac{\pi}{2}} \cos^2 x \, dx}_{= \frac{\pi}{4}.} = \int_0^{\frac{\pi}{2}} 1 \, dx = \frac{\pi}{2}.$$